Eigenvalues & Eigenvectors

Definition

Let A be an $n \times n$ matrix. The constant λ is an **eigenvalue** of A if there is a **nonzero** vector $\vec{v}$ (**eigenvector**) such that

$$A\vec{v} = \lambda\vec{v} \qquad \textbf{(5)}$$

Q1: Is $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ an eigenvector of $\begin{bmatrix} -3 & 1 \\ -3 & 8 \end{bmatrix}$? If yes, find the corresponding eigenvalue.

For the answer to be yes, there must be a constant λ such that

$$\begin{bmatrix} -3 & 1 \\ -3 & 8 \end{bmatrix}\begin{pmatrix} 1 \\ 4 \end{pmatrix} = \lambda\begin{pmatrix} 1 \\ 4 \end{pmatrix}.$$

$$\begin{bmatrix} -3 & 1 \\ -3 & 8 \end{bmatrix}\begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} -3(1)+1(4) \\ -3(1)+8(4) \end{pmatrix} = \begin{pmatrix} 1 \\ 29 \end{pmatrix} \neq \lambda\begin{pmatrix} 1 \\ 4 \end{pmatrix}$$ **Answer: No**

Q2: Is $\lambda = 8$ an eigenvalue of $\begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}$? Justify your answer.

For the answer to be yes, there must be a nonzero vector $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ such that

$$\begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 8\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad \text{or} \quad \begin{aligned} 7v_1 + 3v_2 &= 8v_1 \\ 3v_1 - v_2 &= 8v_2 \end{aligned}$$

which is equivalent to finding nontrivial solutions to the homogeneous system

$$\begin{aligned} v_1 - 3v_2 &= 0 \\ 3v_1 - 9v_2 &= 0 \end{aligned}, \quad \text{whose solutions are } \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 3v_2 \\ v_2 \end{pmatrix} = v_2\begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

Therefore, $\lambda = 8$ is an eigenvalue of the matrix $\begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}$.

Check: $$\begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}\begin{pmatrix} 3v_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} 24v_2 \\ 8v_2 \end{pmatrix} = 8\begin{pmatrix} 3v_2 \\ v_2 \end{pmatrix}$$

Q3: Is $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ an eigenvector of $\begin{bmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & 5 \end{bmatrix}$? If yes, find the corresponding eigenvalue.

For the answer to be yes, there must be a constant λ such that

$$\begin{bmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & 5 \end{bmatrix}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

$$\begin{bmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & 5 \end{bmatrix}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3(1)+6(-2)+7(1) \\ 3(1)+3(-2)+7(1) \\ 5(1)+6(-2)+5(1) \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \\ -2 \end{pmatrix} = -2\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

Answer: Yes, and the corresponding eigenvalue is $\lambda = -2$.

How to find eigenvalues:

Let A be an $n \times n$ matrix. Starting with $A\vec{v} = \lambda\vec{v}$, $A\vec{v} - \lambda\vec{v} = \vec{0}$, $(A - \lambda I_n)\vec{v} = \vec{0}$.
To find the eigenvectors of A, the homogeneous system $(A - \lambda I_n)\vec{v} = \vec{0}$ must have nontrivial solutions, and therefore,

$$\det(A - \lambda I_n) = 0 \qquad \textbf{(6)}$$

Characteristic Polynomial

The characteristic polynomial of A, $P(\lambda)$, is a polynomial of degree n in the variable λ given by

$$P(\lambda) = \det(A - \lambda I_n)$$

Therefore, the eigenvalues of A are roots of the characteristic polynomial.

Example 1

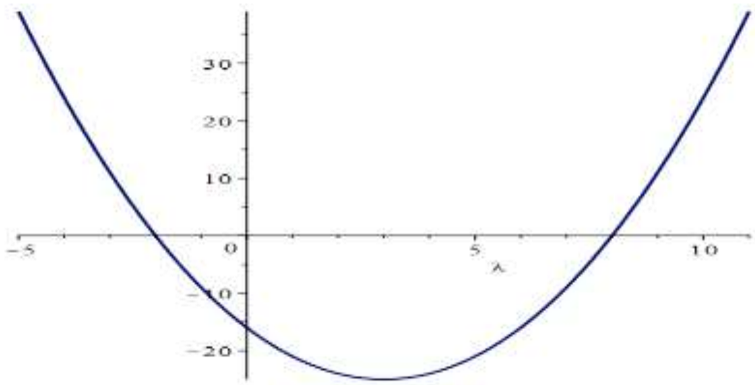
Find the characteristic polynomial and eigenvalues of the matrix

$$A = \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}.$$

$$\det\,(A - \lambda I_2) = \det \begin{bmatrix} 7-\lambda & 3 \\ 3 & -1-\lambda \end{bmatrix} = (7-\lambda)\cdot(-1-\lambda) - 9 = -7-7\lambda+\lambda+\lambda^2-9$$

$$P(\lambda) \ = \ \lambda^2 - 6\lambda - 16$$

$$\lambda^2 - 6\lambda - 16 = (\lambda-8)\cdot(\lambda+2) = 0\ , \quad \lambda_1 = 8\ , \quad \lambda_2 = -2$$



Example 2

Find the characteristic polynomial and eigenvalues of the matrix

$$A = \begin{bmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix}.$$

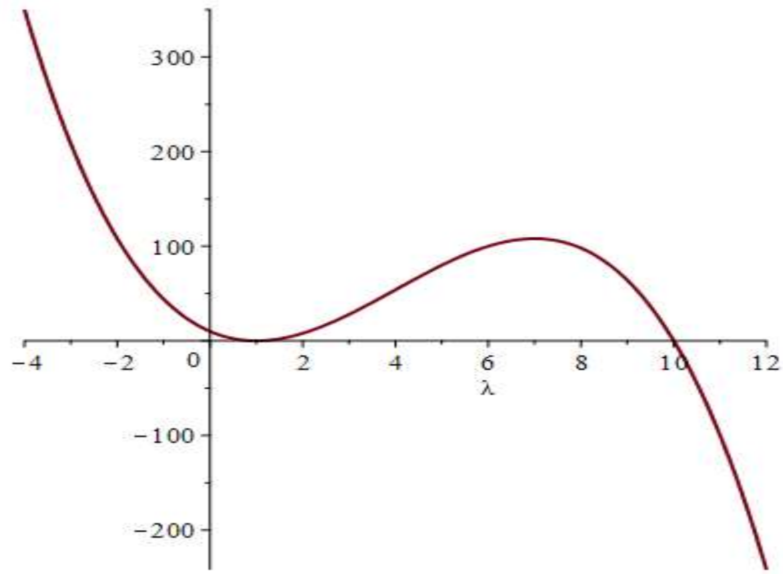
$$P(\lambda) \ = \det\left(A - \lambda I_3\right) = \det \begin{bmatrix} 5-\lambda & 4 & 2 \\ 4 & 5-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{bmatrix} = (5-\lambda)\cdot C_{11} + 4\cdot C_{12} + 2\cdot C_{13}$$

$$= (5-\lambda)\cdot\det\begin{bmatrix} 5-\lambda & 2 \\ 2 & 2-\lambda \end{bmatrix} - 4\cdot\det\begin{bmatrix} 4 & 2 \\ 2 & 2-\lambda \end{bmatrix} + 2\cdot\det\begin{bmatrix} 4 & 5-\lambda \\ 2 & 2 \end{bmatrix}$$

$$= (5-\lambda)\cdot\left[(5-\lambda)(2-\lambda)-4\right] - 4\cdot\left[4(2-\lambda)-4)\right] + 2\cdot\left[8-2(5-\lambda)\right]$$

$$= -\lambda^3 + 12\lambda^2 - 21\lambda + 10 \; = (1-\lambda)^2 \cdot (10-\lambda)$$

Setting $P(\lambda) = 0$ we get $\lambda_1 = 10$ and $\lambda_2 = \lambda_3 = 1$.



Homework

1. Find the characteristic polynomial and eigenvalues of

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

2. Find the characteristic polynomial and all eigenvalues of the matrix

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}.$$

$$P(\lambda) \ = \det\left(C - \lambda I_3\right) = \det\begin{bmatrix} 1-\lambda & 2 & 3 \\ 1 & 2-\lambda & 3 \\ 1 & 2 & 3-\lambda \end{bmatrix} = \cdots$$

How to find eigenvectors:

Recall

To find the eigenvectors of A, we must find nontrivial solutions for the homogeneous system

$$(A - \lambda I_n)\vec{v} = \vec{0} \qquad \textbf{(7)}$$

after replacing λ by the eigenvalues of A, one at a time.

Example 1

Find the eigenvectors of the matrix

$$A = \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}.$$

From an earlier problem we know that $\lambda_1 = 8$ and $\lambda_2 = -2$.

For $\lambda_1 = 8$, $(A - 8I_2) = \begin{bmatrix} -1 & 3 \\ 3 & -9 \end{bmatrix}$, and by solving

$$\begin{bmatrix} -1 & 3 \\ 3 & -9 \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \text{we have}$$

$-v_1 + 3v_2 = 0$, $v_1 = 3v_2$ and $\vec{v}^{\,(1)} = \begin{pmatrix} 3v_2 \\ v_2 \end{pmatrix}$.

For $\lambda_2 = -2$, $(A + 2I_2) = \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}$, and by solving

$$\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \text{we have}$$

$3v_1 + v_2 = 0$, $v_1 = -\dfrac{v_2}{3}$ and $\vec{v}^{\,(2)} = \begin{pmatrix} -\dfrac{v_2}{3} \\ v_2 \end{pmatrix}$.

Example 2

Find the eigenvectors of the matrix

$$A = \begin{bmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix}.$$

From an earlier problem we know that $\lambda_1 = 10$ and $\lambda_2 = \lambda_3 = 1$.

For $\lambda_1 = 10$, $(A - 10I_3) = \begin{bmatrix} -5 & 4 & 2 \\ 4 & -5 & 2 \\ 2 & 2 & -8 \end{bmatrix}$, and by solving

$$\begin{bmatrix} -5 & 4 & 2 \\ 4 & -5 & 2 \\ 2 & 2 & -8 \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \text{we have}$$

$$\left[\begin{array}{ccc|c} -5 & 4 & 2 & 0 \\ 4 & -5 & 2 & 0 \\ 2 & 2 & -8 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} -1 & -1 & 4 & 0 \\ 4 & -5 & 2 & 0 \\ 2 & 2 & -8 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & -4 & 0 \\ 0 & -9 & 18 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 1 & -4 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Solutions: $\vec{v}^{(1)} = \begin{pmatrix} 2v_3 \\ 2v_3 \\ v_3 \end{pmatrix}$

For $\lambda_{1,2} = 1$, $(A - 1I_3) = \begin{bmatrix} 4 & 4 & 2 \\ 4 & 4 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, and by solving

$$\begin{bmatrix} 4 & 4 & 2 \\ 4 & 4 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \text{we get}$$

$$\left[\begin{array}{ccc|c} 4 & 4 & 2 & 0 \\ 4 & 4 & 2 & 0 \\ 2 & 2 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & \frac{1}{2} & 0 \\ 4 & 4 & 2 & 0 \\ 2 & 2 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Solutions: $\begin{pmatrix} -v_2 - \frac{1}{2}v_3 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} -v_2 \\ v_2 \\ 0 \end{pmatrix} + \begin{pmatrix} -\frac{1}{2}v_3 \\ 0 \\ v_3 \end{pmatrix}$

Therefore,

$$\vec{v}^{(2)} = \begin{pmatrix} -v_2 \\ v_2 \\ 0 \end{pmatrix} \quad \text{and} \quad \vec{v}^{(3)} = \begin{pmatrix} -\frac{1}{2}v_3 \\ 0 \\ v_3 \end{pmatrix}.$$

Theorem 5

If λ is an eigenvalue of an $n \times n$ invertible matrix A, then $\dfrac{1}{\lambda}$ is an eigenvalue of A^{-1}.

Proof:

If λ is an eigenvalue of an $n \times n$ invertible matrix A, then there is a vector $\vec{v}$ such that $A\vec{v} = \lambda\vec{v}$. Since A is invertible, we apply A^{-1} on both sides. Therefore,

$$A^{-1}A\vec{v} = A^{-1}\lambda\vec{v} \quad \text{or} \quad \vec{v} = \lambda A^{-1}\vec{v} \quad \text{or} \quad A^{-1}\vec{v} = \frac{1}{\lambda}\vec{v}$$

Theorem 6

If $\lambda = 0$ is an eigenvalue of an $n \times n$ matrix A, then $\det A = 0$.

Proof:

If λ is an eigenvalue of an $n \times n$ invertible matrix A, then $\det\ A - \lambda I_n\ = 0$. If you let $\lambda = 0$ then $\det A = 0$.

Theorem 7

If λ is an eigenvalue of an $n \times n$ matrix A, then λ^2 is an eigenvalue of A^2.

Proof:

If λ is an eigenvalue of an $n \times n$ invertible matrix A, then there is a vector $\vec{v}$ such that $A\vec{v} = \lambda\vec{v}$. If we multiply both sides by A we get,

$$AA\vec{v} = A\lambda\vec{v} \quad \text{or} \quad A^2\vec{v} = \lambda A\vec{v} \quad \text{or} \quad A^2\vec{v} = \lambda \cdot \lambda\vec{v} \quad \text{or} \quad A^2\vec{v} = \lambda^2\vec{v}$$

Homework

1. Find all eigenvectors of A and B.

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

2. Find all eigenvectors of the matrix

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}.$$